Roll No. ....
400 -/-/10/20

JS-259-N

June – July 2022

M. A. / M. Sc. II Semester Examination

MATHEMATICS

PAPER II : LEBESGUE MEASURE AND INTEGRATION

Time 3 Hours] [Max. Marks : Regular 85 / Private 100
[Min. Marks : Regular 28 / Private 33

Note : This question paper is meant for all Regular and Private students. Answer all five questions. Attempt any two parts from each question. All questions carry equal marks. The blind candidates will be given 60 minutes extra time.

1. (a) Let $m^*(E)$ be finite. Show that E is measurable if anal only if given $\in > 0$ there is a finite union U of open intervals such that $m^*(U \Delta E) < \in$.

   (b) Define G_δ, F_σ sets. Show that there are borel sets. Further, show that each open interval is an F_σ set and each closed interval is a G_δ-set.

   (c) Show that there exists a non-measurable set in [0, 1). Further, show that if A is any set in $\mathbb{R}$ with $m^*(A) > 0$ then there is a non-measurable set $E \subset A$.

2. (a) State and prove Lusin's Theorem.

   (b) State and prove Egoroff's Theorem. Show that finiteness of a measure of the set in the statement of the theorem is an essential condition.

   (c) State four equivalent conditions for a real valued function f to be measurable. Prove that these four conditions are equivalent. Further, show that these four statements imply that for each extended real number α the set $\{x \mid f(x) = \alpha\}$ is measurable. Next, show that converse is not true.

3. (a) State and prove Monotone Convergence Theorem. Further, also prove that for a sequence of non-negative measurable functions $\{f_n\}$ which converges to f such that $f_n \leq f$ for all n,

   $$\int f = \lim_{n \to \infty} \int f_n$$

   Next, show that Monotone Convergence Theorem need not hold for decreasing sequence of functions.

   (b) Let f and g be integrable over E and let C be a constant. Then show that :

   (i) The function cf is integrable and

   $$\int_E cf = c\int_E f.$$

   (ii) The function $f + g$ is integrable over E and

   $$\int_E f + g = \int_E f + \int_E g$$

   (iii) If $f \leq g$ a. e. then $\int_E f \leq \int_E g$.

   (c) Define Vitali Cover. Give an example. State and prove Vitali Covering Lemma.

4. (a) If f is bounded and measurable on $[a, b]$ and

   $$F(x) = \int_a^x f(t)\, dt + F(a)$$

   then prove that $F'(x) = f(x)$ a. e. on $[a, b]$. Further, show that the result is true for integrable functions on $[a, b]$.

   (b) (i) If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a. e. then show that f is constant.

   (ii) Show that a function F is an indefinite integral if and only if it is absolutely continuous.

   (c) State and prove Minkowski and Hölder Inequalities.

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5. (a) Given $f \in L_p, 1 \leq p \leq \infty$ and $\in > 0$ show that there is a bounded measurable function f_M with $|f_M| \leq M$ and $||f - f_M|| < \in$. Further, for $1 < p \leq \infty$ show that there is a step function ϕ and a continuous function ψ such that $||f - \phi|| < \in$ and $||f - \psi|| < \in$.

(b) Prove that for each function g in L_q defines a bounded linear functional F on L_p by

$$F(f) = \int fg$$

With $||F|| = ||g||_q$. Further, if g is an integrable function on [0, 1] and suppose there is a constant M such that :

$$\left|\int fg\right| \leq M\,||f||_p$$

for all bounded measurable functions f then show that $g \in L_q$ and $||g||_q \leq M$.

(c) State and prove Riesz-Representation Theorem.
